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Basic gerbe over non-simply connected compact groups

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Abstract

We present an explicit construction of the basic bundle gerbes with connection over all connected compact simple Lie groups. These are geometric objects that appear naturally in the Lagrangian approach to the WZW conformal field theories. Our work extends the recent construction of Meinrenken [The basic gerbe over a compact simple Lie group, L'Enseignement Mathematique, in press. arXiv:math. DG/0209194] restricted to the case of simply connected groups.

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1. Introduction

Bundle gerbes [4,10,13,14] are geometric objects glued from local inputs with the use of transition data forming a 1-cocycle of line bundles. In a version equipped with connection, they found application in the Lagrangian approach to string theory where they permit to treat in an intrinsically geometric way the Kalb-Ramond 2-form fields B that do not exist globally [1,7,9,16]. One of the simplest situations of that type involves group manifolds G when the (local) B field satisfies $\mathrm{d}B = H$ with $H = (\kappa/12\pi)\,\mathrm{tr}\,(g^{-1}\,\mathrm{d}g)^3$. Such B fields appear in the WZW conformal field theories of level κ and the related coset models [8,18]. Construction of the corresponding gerbes allows a systematic Lagrangian treatment of such models, in particular, of the conformal boundary conditions corresponding to open string branes. This

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was discussed in a detailed way in Ref. [9]. The abstract framework was illustrated there by the example of the $SU(N)$ group and of groups covered by $SU(N)$. Here we extend the recent construction [12] of the basic gerbe with $\mathsf{k} = 1$ on all simple, connected and simply connected compact groups G to the case of non-simply connected groups $G' = G/Z$ where Z is a subgroup of the center of G, see [2,3] for other constructions of gerbes on Lie groups. Similarly as in the case of groups covered by $SU(N)$, the obstruction that prevents the basic gerbe on G from descending to G' is a cohomology class $[U] \in H^3(Z, U(1))$. The basic gerbes on group G' correspond to the level k equal to the smallest positive integer such that $[U^{\mathsf{K}}] = 1$. Their pullback to G is the kth power of the basic gerbe on G. The 2-cochains u such that $\delta u = U^{\mathsf{K}}$ provide the essential data for their construction. We explicitly calculate U, k and u for all G from the Cartan series and all Z. The constraints on the level k that we find here where first worked out in Ref. [5] by examining when the 3-forms H on the group G' have periods in $2\pi\mathbb{Z}$. This is a necessary and sufficient condition for existence of the corresponding gerbe on G'. The aim of [5] was to calculate the toroidal partition functions of the WZW models with groups G' as targets. The approach based on the modular invariance of toroidal partition functions [11] reproduced later the same results. The present construction opens the possibility to extend to the other non-simply connected groups the geometric classification of branes and the calculatation of the corresponding annular partition functions worked out in Ref. [9] for the WZW models with groups covered by $SU(N)$, see [6] for a different approach to the construction of annular partition function in the WZW models.

2. Basic gerbe on simply connected compact groups [12]

We refer the reader to Ref. [13] for an introduction to bundle gerbes with connection, to [14] for the notion of stable isomorphism of gerbes (employed below in accessory manner) and to [9] for a discussion of the relevance of the notions for the WZW models of conformal quantum field theory. For completeness, we shall only recall here the basic definition [13]. For $\pi : Y \mapsto M$, let

$$Y^{[n]} = Y \times_M Y \cdots \times_M Y = \{(y_1, \ldots, y_n) \in Y^n | \pi(y_1) = \cdots = \pi(y_n)\} \tag{2.1}$$

denote the n-fold fiber product of Y, $\pi^{[n]}$ the obvious map from $Y^{[n]}$ to M and $p_{n_1 \cdots n_k}$ the projection of $(y_1, \ldots, y_n)$ to $(y_{n_1}, \ldots, y_{n_k})$. Let H be a closed 3-form on manifold M.

Definition. A bundle gerbe $\mathcal{G}$ with connection (shortly, a gerbe) of curvature H over M is a quadruple (Y, B, L, μ), where

1. Y is a manifold provided with a surjective submersion $\pi : Y \to M$.
2. B is a 2-form on Y such that

$$\mathrm{d}B = \pi^* H, \tag{2.2}$$

3. L is a hermitian line bundle with a (unitary) connection over $Y^{[2]}$ with the curvature form:

$$F = p_2^* B - p_1^* B. \tag{2.3}$$

4. $\mu : p_{12}^{*}L \otimes p_{23}^{*}L \to p_{13}^{*}L$ is an isomorphism between the line bundles with connection over $Y^{[3]}$ such that over $Y^{[4]}$:

$$\mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu). \tag{2.4}$$

The 2-form B is called the curving of the gerbe. The isomorphism μ defines a structure of a groupoid on L with the bilinear product $\mu : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \to L_{(y_1,y_3)}$.

Throughout this paper, G will denote a simple connected and simply connected compact Lie group, $\mathbf{g}$ its Lie algebra[2] and tr the linear functional on the enveloping algebra $U(\mathbf{g})$ proportional to the trace in the adjoint representation appropriately normalized (see below). In Ref. [12], an explicit and elegant construction of a gerbe on G with curvature $H = (\kappa/12\pi)\,\mathrm{tr}\,(g^{-1}\,\mathrm{d}g)^3$ for the minimal value of $\kappa > 0$ was given. This gerbe, named "basic", corresponds to $\kappa = 1$ in our normalization of tr. It is unique up to stable isomorphisms. We re-describe here the construction of [12] in a somewhat more concrete and less elegant terms (Ref. [12] constructed the gerbe equivariant w.r.t. the adjoint action; we skip here the higher order equivariant corrections).

Let us first collect some simple facts and notations employed in the sequel. Let

$$\mathbf{g}^{\mathbb{C}} = \mathbf{t}^{\mathbb{C}} \oplus \left(\bigoplus_{\alpha\in\Delta} \mathbb{C}e_\alpha \right) \tag{2.5}$$

be the root decomposition of the complexification of $\mathbf{g}$, with $\mathbf{t}$ standing for the Cartan algebra and Δ for the set of roots in the dual of $\mathbf{t}$. We identify in the standard way $\mathbf{t}$ and $\mathbf{g}$ with their duals using the ad-invariant bilinear form $\mathrm{tr}\,XY$ on $\mathbf{g}$. The normalization of tr is chosen so that the long roots viewed as elements of $\mathbf{t}$ have length squared 2. Let r be the rank of $\mathbf{g}$ and α_i, $\alpha_i^\vee$, λ_i, $\lambda_i^\vee$, $i = 1, \ldots, r$, be the simple roots, coroots, weights and coweights of $\mathbf{g}$ generating, respectively, the lattices Q, $Q^\vee$, P and $P^\vee$ in $\mathbf{t}$. The roots and coroots satisfy $\alpha = 2\alpha^\vee/\mathrm{tr}(\alpha^\vee)^2$. The highest root $\phi = \sum_{i=1}^{r} k_i\alpha_i = \phi^\vee = \sum_{i=1}^{r} k_i^\vee\alpha_i^\vee$. The dual Coxeter number $h^\vee = \sum_{i=0}^{r} k_i^\vee$, where $k_0 = k_0^\vee = 1$. The space of conjugacy classes in G, i.e. of the orbits of the adjoint action of G on itself, may be identified with the Weyl alcove:

$$\mathcal{A} = \{\tau \in \mathbf{t} | \mathrm{tr}\,\alpha_i\tau \geq 0,\quad i = 1, \ldots, r,\quad \mathrm{tr}\,\phi\tau \leq 1\} \tag{2.6}$$

since every conjugacy class has a single element of the form $\mathrm{e}^{2\pi\mathrm{i}\tau}$ with $\tau \in \mathcal{A}$. Set $\mathcal{A}$ is a simplex with vertices $\tau_i = (1/k_i)\lambda_i^\vee$ and $\tau_0 = 0$. Let

$$\mathcal{A}_0 = \{\tau \in \mathcal{A} | \mathrm{tr}\,\phi\tau < 1\} \quad \text{and} \quad \mathcal{A}_i = \{\tau \in \mathcal{A} | \mathrm{tr}\,\alpha_i\tau > 0\} \quad \text{for } i \neq 0 \tag{2.7}$$

and let $\mathcal{A}_I = \cap_{i\in I}\mathcal{A}_i$ for $I \subset \{0, 1, \ldots, r\} \equiv R$. We shall denote by G_i the adjoint action stabilizer of $\mathrm{e}^{2\pi\mathrm{i}\tau_i}$:

$$G_i = \{\gamma \in G | \gamma\,\mathrm{e}^{2\pi\mathrm{i}\tau_i}\gamma^{-1} = \mathrm{e}^{2\pi\mathrm{i}\tau_i}\} \tag{2.8}$$

and by $\mathbf{g}_i$ its Lie algebra. The complexification of $\mathbf{g}_i$ is

$$\mathbf{g}_i^{\mathbb{C}} = \mathbf{t}^{\mathbb{C}} \oplus \left(\bigoplus_{\alpha\in\Delta_i} \mathbb{C}e_\alpha \right), \tag{2.9}$$

[2] We use the physicist convention with $\mathrm{e}^{\mathrm{i}X} \in G$ for $X \in \mathbf{g}$ and $\mathrm{i}[X, Y] \in \mathbf{g}$ for $X, Y \in \mathbf{g}$.

where Δ_i is composed of roots α such that $\mathrm{tr}\,\tau_i\alpha \in \mathbb{Z}$. For $i = 0$, $G_0 = G$ and $\mathbf{g}_0 = \mathbf{g}$. For $i \neq 0$, $\mathbf{g}_i$ is the simple Lie algebra with simple roots α_j, $j \neq i$, and $-\phi$. Its simple coweights are $k_j(\tau_j - \tau_i)$, $j \neq i$ and $-\tau_i$ and they generate the coweight lattice $P_i^\vee$ of $\mathbf{g}_i$.

The main complication in the construction of the basic gerbe over general compact simply connected groups is that the stabilizers G_i are connected but, unlike for $SU(N)$, they are not necessarily simply connected. We shall denote by $\tilde{G}_i$ their universal covers. $G_i = \tilde{G}_i/\mathcal{Z}_i$, where $\mathcal{Z}_i$ is the subgroup of the center of $\tilde{G}_i$. $\mathcal{Z}_i$ is composed of elements of the form $\mathrm{e}_i^{2\pi \mathrm{i} p}$ with $p \in Q^\vee$ and e_i standing for the exponential map from $\mathrm{i}\mathbf{g}_i$ to $\tilde{G}_i$. Since τ_i is also a weight of $\mathbf{g}_i$, it defines a character χ_i on the Cartan subgroup $\tilde{T}_i$ of $\tilde{G}_i$, and hence also on $\mathcal{Z}_i$, by the formula:

$$\chi_i(\mathrm{e}_i^{2\pi \mathrm{i}\tau}) = \mathrm{e}^{2\pi \mathrm{i}\,\mathrm{tr}\,\tau_i\tau}. \tag{2.10}$$

The characters χ_i may be used to define flat line bundles $\hat{L}_i$ over groups G_i by setting

$$\hat{L}_i = (\tilde{G}_i \times \mathbb{C})/\underset{i}{\sim} \tag{2.11}$$

with the equivalence relation:

$$(\tilde{\gamma}, u)\underset{i}{\sim}(\tilde{\gamma}\zeta, \chi_i(\zeta)^{-1}u) \tag{2.12}$$

for $\zeta \in \mathcal{Z}_i$. Note that the left and right action of $\tilde{G}_i$ on itself defines an action of $\tilde{G}_i$ by automorphisms of $\hat{L}_i$ preserving the flat structure. The circle subbundle of $\hat{L}_i$ forms under the multiplication induced by the point-wise one in $\tilde{G}_i \times U(1)$ a central extension $\hat{G}_i$ of G_i. These extensions were a centerpiece of the construction of Ref. [12].

For $I \subset R$ with more than one element, one defines subgroups $G_I \subset G$ as the adjoint action stabilizers of elements $\mathrm{e}^{2\pi\mathrm{i}\tau}$ with τ in the open simplex in $\mathcal{A}$ generated by vertices τ_i, $i \in I$ (G_I does not depend on the choice of τ). In general, $G_I \neq \cap_{i\in I} G_i$. To spare on notation, we shall write $G_{\{i,j\}} = G_{ij}$ with $G_{ii} = G_i$, etc. Let $\mathbf{g}_I$ be the Lie algebra of G_I and $\tilde{G}_I$ its universal cover such that $G_I = \tilde{G}_I/\mathcal{Z}_I$. For $J \supset I$, $G_J \subset G_I$ and the inclusion $\mathbf{g}_J \subset \mathbf{g}_I$ induces the homomorphisms of the universal covers:

$$\begin{array}{ccc} \tilde{G}_J & \rightarrow & \tilde{G}_I \\ \downarrow & & \downarrow \\ G_J & \subset & G_I \end{array} \tag{2.13}$$

which map $\mathcal{Z}_J$ in $\mathcal{Z}_I$. G_R is equal to the Cartan subgroup T of G so that $\tilde{G}_R = \mathbf{t}$ and for each I one has a natural homomorphism:

$$\mathbf{t} \overset{\mathrm{e}_I^{2\pi\mathrm{i}\cdot}}{\longrightarrow} \tilde{G}_I \tag{2.14}$$

that maps onto a commutative subgroup $\tilde{T}_I$ covering $T \subset G_I$ and sends the coroot lattice $Q^\vee$ onto $\mathcal{Z}_I$. Let

$$a_{ij} = i\,\mathrm{tr}\,(\tau_j - \tau_i)(\gamma^{-1}\,\mathrm{d}\gamma) \tag{2.15}$$

be a one form on G_{ij}. It is easy to see that a_{ij} is closed. Indeed,

$$\mathrm{d}a_{ij} = i\,\mathrm{tr}\,(\tau_i - \tau_j)(\gamma^{-1}\,\mathrm{d}\gamma)^2 = 0, \tag{2.16}$$

where the last equality follows from the easy to check fact that the adjoint action of the Lie algebra $\mathbf{g}_{ij}$ (and, hence, also of G_{ij}) preserves $\tau_i - \tau_j$. Let χ_{ij} be a $U(1)$-valued function on the covering group $\tilde{G}_{ij}$ such that $\mathrm{i}\chi_{ij}^{-1}\,\mathrm{d}\chi_{ij}$ is the pullback of a_{ij} to $\tilde{G}_{ij}$ and that $\chi_{ij}(1) = 1$. Explicitly,

$$\chi_{ij}(\tilde{\gamma}) = \exp\left[\frac{1}{i}\int_{\tilde{\gamma}} a_{ij}\right], \tag{2.17}$$

where $\tilde{\gamma}$ is interpreted as a homotopy class of paths in G_{ij} starting from 1. It is easy to see that χ_{ij} defines a one-dimensional representation of $\tilde{G}_{ij}$:

$$\chi_{ij}(\tilde{\gamma}\tilde{\gamma}') = \chi_{ij}(\tilde{\gamma})\chi_{ij}(\tilde{\gamma}') \tag{2.18}$$

and that for $\tilde{\gamma} \in \tilde{G}_{ijk}$ that may be also viewed as an element of $\tilde{G}_{ij}$, $\tilde{G}_{jk}$ and $\tilde{G}_{ik}$, see diagram (2.13):

$$\chi_{ij}(\tilde{\gamma})\chi_{jk}(\tilde{\gamma}) = \chi_{ik}(\tilde{\gamma}). \tag{2.19}$$

As may be easily seen from the definition (2.17):

$$\chi_{ij}(\mathrm{e}_{ij}^{2\pi\mathrm{i}\tau}) = \mathrm{e}^{2\pi\mathrm{i}\,\mathrm{tr}\,(\tau_j - \tau_i)\tau} = \chi_j(\mathrm{e}_j^{2\pi\mathrm{i}\tau})\chi_i(\mathrm{e}_i^{2\pi\mathrm{i}\tau})^{-1} \tag{2.20}$$

for $\tau \in \mathbf{t}$, see Eq. (2.10). In particular, for $\zeta \in \mathcal{Z}_{ij}$:

$$\chi_{ij}(\zeta) = \chi_j(\zeta)\chi_i(\zeta)^{-1}, \tag{2.21}$$

where on the right-hand side, ζ is embedded into $\mathcal{Z}_i$ and $\mathcal{Z}_j$ using the homomorphisms (2.13).

The construction of the basic gerbe $\mathcal{G} = (Y, B, L, \mu)$ over group G described in [12] uses a specific open covering $(\mathcal{O}_i)$ of G, where

$$\mathcal{O}_i = \{h\,\mathrm{e}^{2\pi\mathrm{i}\tau}h^{-1} | h \in G, \quad \tau \in \mathcal{A}_i\}. \tag{2.22}$$

Over sets $\mathcal{O}_i$, the closed 3-form H becomes exact. More concretely, the formulae:

$$B_i = \frac{1}{4\pi}\,\mathrm{tr}\,(h^{-1}\,\mathrm{d}h)\,\mathrm{e}^{2\pi\mathrm{i}\tau}(h^{-1}\,\mathrm{d}h)\,\mathrm{e}^{-2\pi\mathrm{i}\tau} + \mathrm{i}\,\mathrm{tr}\,(\tau - \tau_i)(h^{-1}\,\mathrm{d}h)^2 \tag{2.23}$$

define smooth 2-forms on $\mathcal{O}_i$ such that $\mathrm{d}B_i = H$. More generally, let $\mathcal{O}_I = \cap_{i\in I}\mathcal{O}_i$. Since the elements $\mathrm{e}^{2\pi\mathrm{i}\tau}$ with $\tau \in \mathcal{A}_I$ have the adjoint action stabilizers contained in G_I, the maps

$$\mathcal{O}_I \ni g = h\,\mathrm{e}^{2\pi\mathrm{i}\tau}h^{-1} \stackrel{\rho_I}{\to} hG_I \in G/G_I \tag{2.24}$$

are well defined. They are smooth [12]. They will play an important role below. On the double intersections $\mathcal{O}_{ij}$:

$$B_j - B_i = i\,\mathrm{tr}\,(\tau_i - \tau_j)(h^{-1}\,\mathrm{d}h)^2 \tag{2.25}$$

are closed 2-forms but, unlike in the case of the $SU(N)$ (and $Sp(2N)$) groups, their periods are not in $2\pi\mathbb{Z}$, in general. As a result, they are not curvatures of line bundles over $\mathcal{O}_{ij}$. It is here that the general case departs from the $SU(N)$ one as described in [9] where it was enough to take $Y = \sqcup\mathcal{O}_i$.

Let P_I be the pulback by ρ_I of the principal G_I-bundle $G \to G/G_I$, i.e.

$$P_I = \{(g, h) \in \mathcal{O}_I \times G | \rho_I(g) = hG_I\} \tag{2.26}$$

with the natural projection π_I on $\mathcal{O}_I$ and the action of G_I given by the right multiplication of h. Following [12], we set $Y_i = P_i$ and

$$Y = \bigsqcup_{i=0,\ldots,r} Y_i \tag{2.27}$$

with the projection $\pi : Y \to G$ that restricts to π_i on each Y_i. The curving 2-form B on Y is defined by setting

$$B|_{Y_i} = \pi_i^* B_i. \tag{2.28}$$

Clearly, $\mathrm{d}B = \pi^* H$ as required. Let us note that we may identify

$$Y^{[n]} \cong \bigsqcup_{\substack{(i_1,\ldots,i_n)\\ i_m=0,\ldots,r}} Y_{i_1\cdots i_n}, \tag{2.29}$$

where

$$Y_{i_1\cdots i_n} = \hat{Y}_{i_1\cdots i_n}/G_I, \quad \hat{Y}_{i_1\cdots i_n} = P_I \times G_{i_1} \times \cdots \times G_{i_n} \tag{2.30}$$

for $I = \{i_1, \ldots, i_n\}$ with G_I acting diagonally on $\hat{Y}_{i_1\cdots i_n}$ by the right multiplication. The identification assigns to the G_I-orbit of $((g, h), \gamma_1, \ldots, \gamma_n)$ the element $(y_1, \ldots, y_n) \in Y_{i_1} \times \cdots \times Y_{i_n}$ with $y_m = (g, h\gamma_m^{-1})$.

We are left with the construction of the line bundle with connection L over $Y^{[2]}$ and of the groupoid product μ. Denote by $\hat{L}$ the trivial line bundle $P_{ij} \times \mathbb{C}$ over P_{ij} with the connection form:

$$A_{ij} = \mathrm{i}\,\mathrm{tr}\,(\tau_j - \tau_i)(h^{-1}\,\mathrm{d}h) \tag{2.31}$$

(recall that the elements of P_{ij} are pairs (g, h) with $\rho_{ij}(g) = hG_{ij}$). Let $\hat{L}_{ij}$ be the exterior tensor product of the line bundle $\hat{L}$ over P_{ij} with the flat line bundles $\hat{L}_i^{-1}$ on G_i and $\hat{L}_j$ on G_j, see Eq. (2.11). In other words,

$$\hat{L}_{ij} = \hat{p}^*\hat{L} \otimes \hat{p}_i^*\hat{L}_i^{-1} \otimes \hat{p}_j^*\hat{L}_j, \tag{2.32}$$

where $\hat{p}$, $\hat{p}_i$ and $\hat{p}_j$ are the projections from $\hat{Y}_{ij}$ to P_{ij}, G_i and G_j, respectively. Explicitly, the elements of $\hat{L}_{ij}$ may be represented by the pairs $((g, h), [\tilde{\gamma}, \tilde{\gamma}', u]_{ij})$ with the equivalence classes corresponding to the relation:

$$(\tilde{\gamma}, \tilde{\gamma}', u) \underset{ij}{\sim} (\tilde{\gamma}\zeta, \tilde{\gamma}'\zeta', \chi_i(\zeta)\chi_j(\zeta')^{-1}u) \tag{2.33}$$

for $\tilde{\gamma} \in \tilde{G}_i$, $\tilde{\gamma}' \in \tilde{G}_j$, $u \in \mathbb{C}$, $\zeta \in \mathcal{Z}_i$ and $\zeta' \in \mathcal{Z}_j$.

We shall lift the action of G_{ij} on $\hat{Y}_{ij}$ to the action on $\hat{L}_{ij}$ by automorphisms and shall set

$$L|_{Y_{ij}} = \hat{L}_{ij}/G_{ij} \equiv L_{ij}. \tag{2.34}$$

First note that $\tilde{G}_{ij}$ acts on $\hat{L}_{ij}$ by

$$((g,h),[\tilde{\gamma},\tilde{\gamma}',u]_{ij}) \mapsto ((g,h\gamma''),[\tilde{\gamma}\tilde{\gamma}'',\tilde{\gamma}'\tilde{\gamma}'',\chi_{ij}(\tilde{\gamma}'')^{-1}u]_{ij}), \tag{2.35}$$

where $\tilde{\gamma}'' \in \tilde{G}_{ij}$, γ'' is the projection of $\tilde{\gamma}''$ to G_{ij} and χ_{ij} is given by Eq. (2.17). Due to the relations (2.18) and (2.21), $\mathcal{Z}_{ij} \subset \tilde{G}_{ij}$ acts trivially so that the maps (2.35) define the right action of G_{ij} on $\hat{L}_{ij}$. The relation between 1-form A_{ij} and χ_{ij} implies that the connection on $\hat{L}_{ij}$ descends to the quotient line bundle L_{ij}. Note that the curvature of L_{ij} is given by the closed 2-form F_{ij} on Y_{ij} that pulled back to $\hat{Y}_{ij}$ becomes

$$\hat{F}_{ij} = i\,\mathrm{tr}\,(\tau_i - \tau_j)(h^{-1}\,\mathrm{d}h)^2. \tag{2.36}$$

Let p_i and p_j denote the natural projections of Y_{ij} on Y_i and Y_j, respectively. The required relation:

$$p_j^*\pi_j^*B_j - p_i^*\pi_i^*B_i = F_{ij} \tag{2.37}$$

between the curving 2-form and the curvature of L_{ij} follows from the comparison of Eqs. (2.23) and (2.36) with the use of the relation $\rho_{ij}(g) = hG_{ij}$. For $i = j$, line bundle L_{ij} is flat.

We still have to define the groupoid product μ in the line bundles over

$$Y^{[3]} = \bigsqcup_{(i,j,k)} Y_{ijk}, \tag{2.38}$$

see relation (2.29). Let $((g,h),\gamma,\gamma',\gamma'')G_{ijk} \in Y_{ijk}$, where $g \in \mathcal{O}_{ijk}$, $h \in G$ with $\rho_{ijk}(g) = hG_{ijk}$, and where $\gamma \in G_i$, $\gamma' \in G_j$, $\gamma'' \in G_k$. The elements in the corresponding fibers of L_{ij}, L_{jk} and L_{ik} may be defined now as the G_{ijk}-orbits since h is defined by $g \in \mathcal{O}_{ijk}$ up to right multiplication by elements of G_{ijk}. Let

$$\ell_{ij} = ((g,h),[\tilde{\gamma},\tilde{\gamma}',u]_{ij})G_{ijk} \in L_{ij}, \qquad \ell_{jk} = ((g,h),[\tilde{\gamma}',\tilde{\gamma}'',u']_{jk})G_{ijk} \in L_{jk},$$
$$\ell_{ik} = ((g,h),[\tilde{\gamma},\tilde{\gamma}'',uu']_{ik})G_{ijk} \in L_{ik}.$$

One sets

$$\mu(\ell_{ij} \otimes \ell_{jk}) = \ell_{jk}. \tag{2.39}$$

It is easy to see that the right-hand side is well defined. Checking that μ preserves the connection and is associative over $Y^{[4]} = \sqcup_{(i,j,k,l)} Y_{ijkl}$ is also straightforward (the latter is done by rewriting the line bundle elements as G_{ijkl}-orbits).

For $\mathrm{K} \in \mathbb{Z}$, the powers $\mathcal{G}^{\mathrm{K}}$ of the basic gerbe may be constructed the same way by simply exchanging the characters χ_i and homomorphisms χ_{ij} by their Kth powers and by multiplying the connection forms, curvings and curvatures by K. Below, we shall use the notation $[\cdots]_i^{\mathrm{K}}$ and $[\cdots]_{ij}^{\mathrm{K}}$ for the corresponding equivalence classes with such modifications.

3. Basic gerbe on compact non-simply connected groups

Let G be, as before, a connected simply connected simple compact Lie group and let Z be a (non-trivial) subgroup of its center. Let H' be the 3-form on the non-simply connected group $G' = G/Z$ that pulls back to the 3-form $H = (1/12\pi)\,\mathrm{tr}\,(g^{-1}\,\mathrm{d}g)^3$ on G. We shall construct in this section a gerbe $\mathcal{G}' = (Y', B', L', \mu')$ over group G' with curvature $\kappa H'$, where the level κ takes the lowest positive (integer) value for which a gerbe with curvature $\kappa H'$ exists. Such a "basic" gerbe is unique up to stable isomorphisms in all cases except for $G' = SO(4n)/\mathbb{Z}_2$ where there are two non-stably isomorphic basic gerbes, both covered by our construction.

3.1. Some group Z cohomology

We shall need some cohomological construction related to the subgroup Z of the center (for a quick résumé of discrete group cohomology, see Appendix A of [9]).

Group Z acts on the Weyl alcove $\mathcal{A}$ in the Cartan algebra of G by affine transformations. The action is induced from that on G that maps conjugacy classes to conjugacy classes and it may be defined by the formula:

$$z\,\mathrm{e}^{2\pi\mathrm{i}\tau} = w_z^{-1}\,\mathrm{e}^{2\pi\mathrm{i}z\tau}w_z \tag{3.1}$$

for $z \in Z$ and w_z in the normalizer $N(T) \subset G$ of the Cartan subgroup T. In particular, $z\tau_i = \tau_{zi}$ for some permutation $i \mapsto zi$ of the set $R = \{0, 1, \ldots, r\}$ that induces a symmetry of the extended Dynkin diagram with vertices belonging to R and $k_{zi} = k_i$, $k_{zi}^\vee = k_i^\vee$. Explicitly,

$$z\tau = w_z\tau w_z^{-1} + \tau_{z0}. \tag{3.2}$$

Elements $w_z \in N(T)$ are defined up to multiplication (from the right or from the left) by elements in T, so that their classes ω_z in the Weyl group $W = N(T)/T$ are uniquely defined. The assignment $Z \ni z \overset{\omega}{\mapsto} \omega_z \in W$ is an injective homomorphism. However, one cannot always choose $w_z \in N(T)$ so that $w_{zz'} = w_z w_{z'}$. The T-valued discrepancy:

$$c_{z,z'} = w_z w_{z'} w_{zz'}^{-1} \tag{3.3}$$

satisfies the cocycle condition:

$$(\delta c)_{z,z',z''} \equiv (w_z c_{z',z''} w_z^{-1}) c_{zz',z''}^{-1} c_{z,z'z''} c_{z,z'}^{-1} = 1 \tag{3.4}$$

and defines a cohomology class $[c] \in H^2(Z, T)$ that is the obstruction to the existence of a multiplicative choice of w_z. Class $[c]$ is the restriction to $\omega(Z) \subset W$ of the cohomology class in $H^2(W, T)$ that characterizes up to isomorphisms the extension:

$$1 \to T \to N(T) \to W \to 1 \tag{3.5}$$

which was studied in Ref. [17]. The results of [17] could be used to find the 2-cocycle whose cohomology class characterizes the extension (3.5) and then, by restriction, to calculate c. In practice, we found it simpler to obtain the 2-cocycle c directly, see Section 4.

Let us choose elements $b_{z,z'} \in \mathbf{t}$ such that $c_{z,z'} = \mathrm{e}^{2\pi \mathrm{i} b_{z,z'}}$. Note that they are determined up to the replacements:

$$b_{z,z'} \mapsto b_{z.z'} + w_z a_{z'} w_z^{-1} - a_{zz'} + a_z + q_{z,z'} \tag{3.6}$$

with $a_z \in \mathbf{t}$ describing the change $w_z \mapsto \mathrm{e}^{2\pi \mathrm{i} a_z} w_z$ and $q_{z,z'} \in Q^\vee$. The combination

$$(\delta b)_{z,z',z''} \equiv (w_z b_{z',z''} w_z^{-1}) - b_{zz',z''} + b_{z,z'z''} - b_{z,z'} \tag{3.7}$$

is a 3-cocycle on Z with values in $Q^\vee$. It defines a cohomology class $[\delta b] \in H^3(Z, Q^\vee)$, the Bockstein image of $[c]$ induced by the exact sequence

$$0 \to Q^\vee \to \mathbf{t} \stackrel{\mathrm{e}^{2\pi \mathrm{i}\cdot}}{\longrightarrow} T \to 1. \tag{3.8}$$

Note that the replacements (3.6) do not change the cohomology class $[\delta b]$. Below, we shall employ for $I \subset R$ the lifts

$$\tilde{c}_{z,z'} = \mathrm{e}_I^{2\pi \mathrm{i} b_{z,z'}} \in \tilde{T}_I \subset \tilde{G}_I \tag{3.9}$$

of $c_{z,z'}$ to the subgroups $\tilde{T}_I$, see (2.14). Note that

$$(\delta \tilde{c})_{z,z',z''} \equiv \mathrm{e}_I^{2\pi \mathrm{i}(\delta b)_{z,z',z''}} \tag{3.10}$$

belongs to $\mathcal{Z}_I \subset \tilde{T}_I$.

3.2. *Pushing gerbes $\mathcal{G}^{\mathrm{K}}$ to G'*

The structures introduced in the preceding section behave naturally under the action of Z. We have

$$z\mathcal{A}_I = \mathcal{A}_{zI}, \qquad z\mathcal{O}_I = \mathcal{O}_{zI}, \qquad w_z G_I w_z^{-1} = G_{zI}. \tag{3.11}$$

The adjoint action of w_z maps also $\mathbf{g}_I$ onto $\mathbf{g}_{zI}$ and hence lifts to an isomorphism from $\tilde{G}_I$ to $\tilde{G}_{zI}$ that maps $\mathcal{Z}_I$ onto $\mathcal{Z}_{zI}$ and for which we shall still use the notation $\tilde{\gamma}_I \mapsto w_z \tilde{\gamma}_I w_z^{-1}$. The maps $\mathcal{O}_I \ni g \mapsto zg \in \mathcal{O}_{zI}$ may be lifted to the ones

$$P_I \ni y = (g, h) \mapsto zy = (zg, h w_z^{-1}) \in P_{zI} \tag{3.12}$$

of the principal bundles P_I. Note that if $c_{z,z'} \neq 1$ then the lifts do not compose.

Proceeding to construct the basic gerbe $\mathcal{G}' = (Y', B', L', \mu')$ over group G', we shall set

$$Y' = Y = \bigsqcup_{i=0,\ldots,r} Y_i, \qquad B'|_{Y_i} = \kappa\, \pi_i^* B_i, \tag{3.13}$$

where, as before, $Y_i = P_i$ but Y' is taken with the natural projection π' on G'. Note that a sequence $(y, y', \ldots, y^{(n-1)})$ belongs to $Y'^{[n]}$ if $\pi(y) = z\pi(y') = \cdots = zz' \cdots z^{(n-2)} \pi(y^{(n-1)})$ for some $z, z', \ldots, z^{(n-2)} \in Z$. Then

$$(y, zy', \ldots, z(z'(\cdots(z^{(n-2)} y^{(n-1)}) \cdots))) \in Y^{[n]} \tag{3.14}$$

and we may identify

$$Y'^{[n]} \cong \bigsqcup_{(z,z',\ldots,z^{(n-2)})\in Z^{n-1}} Y^{[n]}. \tag{3.15}$$

Let L' be the line bundle on $Y'^{[2]}$ that restricts to L^{K} on each component $Y^{[2]}$ in the identification (3.15), i.e. to L^{K}_{ij} on $Y_{ij} \subset Y^{[2]}$. Since $z^* B_{zi} = B_i$ under the pullback by the maps $\mathcal{O}_i \ni g \mapsto zg \in \mathcal{O}_{zi}$, the curvature F' of L' satisfies the required relation:

$$F' = p_2'^* B' - p_1'^* B', \tag{3.16}$$

where, as usual, p'_1 and p'_2 are the projections in $Y'^{[2]}$ on the first and the second factor.

It remains to define the groupoid multiplication μ'. Let $(y, y', y'') \in Y'^{[3]}$ be such that $(y, zy', z(z'y'')) \in Y_{ijk} \subset Y^{[3]}$. We may then write

$$y = (g, h\gamma^{-1}), \qquad zy' = (g, h\gamma'^{-1}), \qquad z(z'y'') = (g, h\gamma''^{-1}) \tag{3.17}$$

with $g \in \mathcal{O}_{ijk}$ and $h \in G$ such that $\rho_{ijk}(g) = hG_{ijk}$ and with $\gamma \in G_i$, $\gamma' \in G_j$, $\gamma'' \in G_k$. This permits to identify $(y, zy', z(z'y''))$ with $((g, h), \gamma, \gamma', \gamma'')G_{ijk}$ according to (2.30). We shall use the notation $i_z \equiv z^{-1}i$, $\gamma_z \equiv w_z^{-1}\gamma w_z \in G_{i_z}$ for $\gamma \in G_i$ and $\tilde\gamma_z \equiv w_z^{-1}\tilde\gamma w_z \in \tilde G_{i_z}$ for $\tilde\gamma \in \tilde G_i$. Note that

$$y' = (z^{-1}g, hw_z\gamma_z'^{-1}), \qquad y'' = ((zz')^{-1}g, hw_z w_{z'}(\gamma''_z)^{-1}_{z'}), \tag{3.18}$$

$$z'y'' = (z^{-1}g, hw_z\gamma_z''^{-1}), \qquad (zz')y'' = (g, h(c^{-1}_{z,z'}\gamma'')^{-1}). \tag{3.19}$$

Employing the explicit description of the line bundles L_{ij} with $\tilde\gamma \in \tilde G_i$ projecting to γ, etc., we take the elements

$$\ell_{ij} = ((g, h), [\tilde\gamma, \tilde\gamma', u]^{\mathrm{K}}_{ij})G_{ijk} \in L^{\mathrm{K}}_{(y,zy')} = L'_{(y,y')}, \tag{3.20}$$

$$\ell_{j_z k_z} = ((z^{-1}g, hw_z), [\tilde\gamma'_z, \tilde\gamma''_z, u']^{\mathrm{K}}_{j_z k_z})G_{i_z j_z k_z} \in L^{\mathrm{K}}_{(y',z'y'')} = L'_{(y',y'')}, \tag{3.21}$$

$$\ell_{ik} = ((g, h), [\tilde\gamma, \tilde c^{-1}_{z,z'}\tilde\gamma'', uu']^{\mathrm{K}}_{ik})G_{ijk} \in L^{\mathrm{K}}_{(y,(zz')y'')} = L'_{(y,y'')}, \tag{3.22}$$

where $\tilde c_{z,z'} \in \tilde G_k$ is given by Eq. (3.9). Then necessarily,

$$\mu'(\ell_{ij} \otimes \ell_{j_z k_z}) = u^{ijk}_{z,z'}\ell_{ik}, \tag{3.23}$$

where $u^{z,z'}_{ijk}$ are numbers in $U(1)$. That the right-hand side of the definition (3.23) does not depend on the choice of the representatives of the classes on the left-hand side follows from the following lemma.

Lemma 1. *For $z \in Z$, $\zeta \in \mathcal{Z}_i$ and $\tilde\gamma \in \tilde G_{ij}$:*

$$\chi_{i_z}(\zeta_z) = \chi_i(\zeta), \qquad \chi_{i_z j_z}(\tilde\gamma_z) = \chi_{ij}(\tilde\gamma). \tag{3.24}$$

Proof. Let $\zeta = \mathrm{e}_i^{2\pi \mathrm{i} p}$ for $p \in Q^\vee$. Then $\zeta_z = w_z^{-1}\zeta w_z = \mathrm{e}_i^{2\pi\mathrm{i} w_z^{-1} p w_z}$ and

$$\chi_{i_z}(\zeta_z) = \mathrm{e}^{2\pi\mathrm{i}\,\mathrm{tr}\, w_z^{-1}(\tau_i-\tau_{z0})w_z w_z^{-1} p w_z} = \mathrm{e}^{2\pi\mathrm{i}\,\mathrm{tr}\,\tau_i p} = \chi_i(\zeta), \tag{3.25}$$

where we used the fact that $\tau_{z0} = \lambda_{z0}$. The second relation in (3.24) follows immediately from the definition (2.17) of χ_{ij} and the identity $\tau_{j_z} - \tau_{i_z} = w_z^{-1}(\tau_j - \tau_i)w_z$. □

3.3. Obstruction class

It remains to find the conditions under which μ' is associative. In Appendix A, we show by an explicit check that associativity of μ' requires that

$$u^{j_z k_z l_z}_{z',z''}(u^{ikl}_{zz',z''})^{-1} u^{ijl}_{z,z'z''}(u^{ijk}_{z,z'})^{-1} = \chi_{kl}(\tilde{c}_{z,z'})^{\mathrm{K}} \chi_l((\delta\tilde{c})_{z,z',z''})^{\mathrm{K}}. \tag{3.26}$$

This provides an extension of the relation (4.6) of [9] obtained for $G = SU(N)$. It may be treated similarly. First, we set

$$u_{z,z'} = u^{(0)(z0)(zz'0)}_{z,z'} \tag{3.27}$$

and observe that, for $i = j_z = k_{zz'} = l_{zz'z''} = 0$, Eq. (3.26) reduces to the cohomological equation:

$$\delta u = U^{\mathrm{K}}, \tag{3.28}$$

where $(\delta u)_{z,z',z''} = u_{z',z''} u^{-1}_{zz',z''} u_{z,z'z''} u^{-1}_{z,z'}$ is the coboundary of the $U(1)$-valued 2-chain on Z and

$$U_{z,z',z''} = \chi_{(zz'0)(zz'z''0)}(\tilde{c}_{z,z'})\chi_{zz'z''0}((\delta\tilde{c})_{z,z',z''}). \tag{3.29}$$

More exactly, with the use of formulae (2.10), (2.20), (3.2), (3.4), (3.9) and (3.10), one obtains

$$\begin{aligned} U_{z,z',z''} &= \mathrm{e}^{2\pi \mathrm{i}\,\mathrm{tr}[(\tau_{zz'z''0} - z\tau_{zz'0})b_{z,z'} + \tau_{zz'z''0}(w_z b_{z',z''} w_z^{-1} - b_{zz',z''} + b_{z,z'z''} - b_{z,z'})]} \\ &= \mathrm{e}^{2\pi \mathrm{i}\,\mathrm{tr}\,[(\tau_{z'z''0} - \tau_{z^{-1}0})b_{z',z''} - \tau_{zz'0} b_{z,z'} - \tau_{zz'z''0}(b_{zz',z''} - b_{z,z'z''})]}. \end{aligned} \tag{3.30}$$

The cohomological equation (3.28) is consistent due to the following lemma.

Lemma 2. *$(U_{z,z',z''})$ defines a $U(1)$-valued 3-cocycle on Z:*

$$(\delta U)_{z,z',z'',z'''} \equiv U_{z',z'',z'''} U^{-1}_{zz',z'',z'''} U_{z,z'z'',z'''} U^{-1}_{z,z',z''z'''} U_{z,z',z''} = 1. \tag{3.31}$$

Besides its cohomology class $[U] \in H^3(Z, U(1))$ does not depend on the choice of the cocycle $(c_{z,z'})$ in the cohomology class $[c] \in H^2(Z, T)$ nor on the choice of $b_{z,z'} \in \mathbf{t}$ such that $c_{z,z'} = \mathrm{e}^{2\pi \mathrm{i} b_{z,z'}}$.

It is enough to analyze the condition (3.28) due to the following lemma.

Lemma 3. *Let $(u_{z,z'})$ be a solution of* Eq. (3.28). *Then*

$$u^{ijk}_{z,z'} = \chi_{k(zz'0)}(\tilde{c}_{z,z'})^{-\mathrm{K}} u_{z,z'}, \tag{3.32}$$

solves Eq. (3.26).

Remark. The cohomology class $[U] \in H^3(Z, U(1))$ is the obstruction to pushing down the gerbe $\mathcal{G}$ on G to the quotient group G'. That the push-forward of a gerbe by a covering map $M \mapsto M/\Gamma$ requires solving a cohomological problem $U = \delta u$ for a $U(1)$-valued 3-cocycle U on discrete group Γ, with $[U] \in H^3(\Gamma, U(1))$ describing the obstruction class, is a general fact, see [15]. Similar cohomological equation, but in one degree less, with obstruction class in $H^2(\Gamma, U(1))$, describes pushing forward a line bundle. As for the relation (3.32), it is of a geometric origin, as has been explained in [9]: if we choose naturally a stable isomorphism between $\mathcal{G}^{\mathrm{K}}$ and $(z^{-1})^*\mathcal{G}^{\mathrm{K}}$ then the elements $\ell_{ij}^{-1} \otimes \ell_{j_z k_z}^{-1} \otimes \ell_{ik}$ determine flat sections s_{ijk} of a flat line bundle $R^{z,z'}$ on G. Sections s_{ijk} are defined over sets $\mathcal{O}_{ijk}$ and over their intersections $\mathcal{O}_{ijki'j'k'}$, they are related by $s_{i'j'k'} = \chi_{k'k}(\tilde{c}_{z,z'})^{\mathrm{K}} s_{ijk}$.

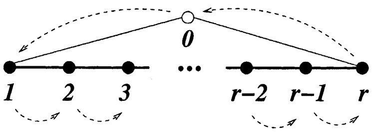
Proofs of Lemmas 2 and 3 may be found in Appendix B. The obstruction cohomology class $[U] \in H^3(Z, U(1))$, explicitly computed in the next section for all groups G', is torsion. The level K of the basic gerbe $\mathcal{G}'$ over G' corresponds to the smallest positive integer for which $[U^{\mathrm{K}}]$ is trivial so that Eq. (3.28) has a solution. In the latter case, different solutions u differ by the multiplication by a $U(1)$-valued 2-cocycle $\tilde{u}$, $\delta\tilde{u} = 1$. If $\tilde{u}$ is cohomologically trivial, i.e. $\tilde{u}_{z,z'} = v_{z'} v_{zz'}^{-1} v_z$, then the modified solution leads to a stably isomorphic gerbe over G'. Whether multiplication of u by cohomologically non-trivial cocycles $\tilde{u}$ leads to stably non-isomorphic gerbes depends on the cohomology group $H^2(G', U(1))$ that classifies different stable isomorphism classes of gerbes over G' with fixed curvature. This is trivial for all simple groups except for $G' = SO(4N)/\mathbb{Z}_2$ when $H^2(G', U(1)) = \mathbb{Z}_2$, see [5].

4. Cocycles *c* and *U*

It remains to calculate the cocycles $c = (c_{z,z'})$, elements $b_{z.z'} \in Q^\vee$ such that $c_{z.z'} = \mathrm{e}^{2\pi \mathrm{i} b_{z,z'}}$ and the cocycles $U = (U_{z,z',z''})$, see Eqs. (3.3) and (3.30), and to solve the cohomological equation (3.28) for all simple, connected, simply connected groups G and all subgroups Z of their center.

4.1. Groups $A_r = SU(r+1)$, $r = 1, 2, \ldots$

The Lie algebra $su(r+1)$ is composed of traceless hermitian $(r+1)\times(r+1)$ matrices. The Cartan algebra may be taken as the subalgebra of diagonal matrices. Let e_i, $i = 1, 2, \ldots, r+1$, denote the diagonal matrices with the j's entry δ_{ij} with $\mathrm{tr}\, e_i e_j = \delta_{ij}$. Roots and coroots of $su(r+1)$ have then the form $e_i - e_j$ for $i \neq j$ and the standard choice of simple roots is $\alpha_i = e_i - e_{i+1}$. The center is $\mathbb{Z}_{r+1}$ and it may be generated by $z = \mathrm{e}^{-2\pi \mathrm{i}\theta}$ with $\theta = \lambda_r^\vee = -e_{r+1} + (1/(r+1)) \sum_{i=1}^{r+1} e_i$. The permutation $zi = i+1$ for $i = 0, 1, \ldots, r-1$, $zr = 0$ generates a symmetry of the extended Dynkin diagram:



The adjoint action of $w_z \in N(T) \subset SU(r+1)$ on the Cartan algebra may be extended to all diagonal matrices by setting

$$w_z e_i w_z^{-1} = \begin{cases} e_1 & \text{if } i = r+1, \\ e_{i+1} & \text{otherwise.} \end{cases} \tag{4.1}$$

It is generated by the product:

$$r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_r} \tag{4.2}$$

of r reflections in simple roots. We may take

$$w_z = \mathrm{e}^{\pi \mathrm{i} r/(r+1)} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{4.3}$$

Setting $w_{z^n} = w_z^n$ for $n = 0, 1, \ldots, r$, we then obtain

$$c_{z^n, z^m} = \begin{cases} 1 & \text{if } n+m \le r, \\ w_z^{r+1} & \text{if } n+m > r. \end{cases} \tag{4.4}$$

Since $w_z^{r+1} = (-1)^r = \mathrm{e}^{2\pi \mathrm{i} X}$ for $X = (r(r+1)/2)\theta$, we may take

$$b_{z^n, z^m} = \begin{cases} 0 & \text{if } n+m \le r, \\ \dfrac{r(r+1)}{2}\theta & \text{if } n+m > r. \end{cases} \tag{4.5}$$

Explicit calculation of the right-hand side of Eq. (3.30) gives

$$U_{z^n, z^{n'}, z^{n''}} = (-1)^{rn''(n+n'-[n+n'])/(r+1)}, \tag{4.6}$$

where $0 \le n, n', n'' \le r$ and for an integer m, $[m] = m \bmod (r+1)$ with $0 \le [m] \le r$.

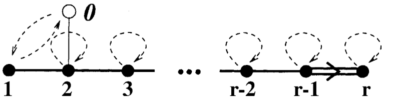
Let $r+1 = N'N''$ and Z be the cyclic subgroup of order N' of the center generated by $z^{N''}$. If N'' is even or N' is odd or κ is even, then the restriction to Z of the cocycle U^{κ} is trivial. In the remaining case of N' even, N'' odd and κ odd it defines a non-trivial class in $H^3(Z, U(1))$. Hence the smallest positive value of the level for which the cohomological equation (3.28) may be solved is

$$\kappa = \begin{cases} 1 & \text{for } N' \text{ odd or } N'' \text{ even,} \\ 2 & \text{for } N' \text{ even and } N'' \text{ odd,} \end{cases} \tag{4.7}$$

in agreement with [9]. For those values of κ, one may take $u_{z^n, z^{n'}} \equiv 1$ as the solution of Eq. (3.28).

4.2. Groups $B_r = \mathrm{Spin}(2r+1)$, $r = 2, 3, \ldots$

The Lie algebra of B_r is $so(2r+1)$. It is composed of imaginary antisymmetric $(2r+1)\times(2r+1)$ matrices. The Cartan algebra may be taken as composed of r blocks $\begin{pmatrix} 0 & -\mathrm{i}t_i \\ \mathrm{i}t_i & 0 \end{pmatrix}$ placed diagonally, with the last diagonal entry vanishing. Let e_i denote the matrix corresponding to $t_j = \delta_{ij}$. With the invariant form normalized so that $\mathrm{tr}\, e_i e_j = \delta_{ij}$, roots of $so(2r+1)$ have the form $\pm e_i \pm e_j$ for $i \neq j$ and $\pm e_i$ and one may choose $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, r-1$ and $\alpha_r = e_r$ as the simple roots. The coroots are $\pm e_i \pm e_j$ for $i \neq j$ and $\pm 2e_i$. The center of $\mathrm{Spin}(2r+1)$ is $\mathbb{Z}_2$ with the non-unit element $z = \mathrm{e}^{-2\pi\mathrm{i}\theta}$ with $\theta = \lambda_1^\vee = e_1$. $SO(2r+1) = \mathrm{Spin}(2r+1)/\{1, z\}$. The permutation $z0 = 1$, $z1 = 0$, $zi = i$ for $i = 2, \ldots, r$ generates a symmetry of the extended Dynkin diagram:



The adjoint action of $w_z \in N(T)$ is given by

$$w_z e_i w_z^{-1} = \begin{cases} -e_1 & \text{if } i = 1, \\ e_i & \text{if } i \neq 1. \end{cases} \tag{4.8}$$

It may be generated by the product:

$$r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_{r-2}} r_{\alpha_{r-1}} r_{\alpha_r} r_{\alpha_{r-1}} \cdots r_{\alpha_2} r_{\alpha_1} \tag{4.9}$$

of $2r-1$ reflections in simple roots. Element w_z may be taken as the lift to $\mathrm{Spin}(2r+1)$ of the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \tag{4.10}$$

in $SO(2r+1)$. Setting also $w_1 = 1$, we infer that

$$c_{1,1} = c_{1,z} = c_{z,1} = 1, \qquad c_{z,z} = w_z^2. \tag{4.11}$$

Since w_z^2 projects to 1 in $SO(2r+1)$, it is equal to 1 or to z. To decide which is the case, we write $w_z = \mathcal{O}\,\mathrm{e}^{2\pi i \mathrm{X}}\mathcal{O}^{-1}$, where $\mathcal{O} \in \mathrm{Spin}(2r+1)$ projects to the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{4.12}$$

in $SO(2r+1)$ and $X = (1/2)\sum_{i=1}^{r} e_r$ so that $\mathrm{e}^{2\pi \mathrm{i} X}$ projects to the matrix:

$$\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{4.13}$$

in $SO(2r+1)$. Now $w_z^2 = 1$ if and only if $2X$ is in the coroot lattice. This happens if r is even. We may then take

$$b_{1,1} = b_{1,z} = b_{z,1} = b_{z,z} = 0 \tag{4.14}$$

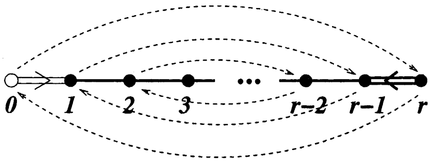
for even r and

$$b_{1,1} = b_{1,z} = b_{z,1} = 0, \quad b_{z,z} = \theta \tag{4.15}$$

for odd r. Here $U_{z^n,z^{n'},z^{n''}} \equiv 1$ for all $0 \le n, n', n'' \le 1$. Hence $\kappa = 1$ and $u_{z^n,z^{n'}} \equiv 1$ solves Eq. (3.28).

4.3. Groups $C_r = Sp(2r)$, $r = 2, 3, \ldots$

This is a group composed of unitary $(2r)\times(2r)$ matrices U such that $U^{\mathrm{T}}\Omega U = \Omega$ for Ω built of r blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ placed diagonally. For $r = 2$, $Sp(4) \cong \mathrm{Spin}(5)$. The Lie algebra $sp(2r)$ of groups D_r is composed of hermitian $(2r)\times(2r)$ matrices X such that ΩX is symmetric. The Cartan subalgebra may be taken as composed of r blocks $\begin{pmatrix} 0 & -\mathrm{i}t_i \\ \mathrm{i}t_i & 0 \end{pmatrix}$ placed diagonally. Let e_i denote the matrix corresponding to $t_j = \delta_{ij}$. With the invariant form normalized so that $\mathrm{tr}\, e_i e_j = 2\delta_{ij}$, roots of $sp(2r)$ have the form $(1/2)(\pm e_i \pm e_j)$ for $i \ne j$ and $\pm e_i$. The simple roots may be chosen as $\alpha_i = (1/2)(e_i - e_{i+1})$ for $i = 1, \ldots, r-1$ and $\alpha_r = e_r$. The coroots are $\pm e_i \pm e_j$ for $i \ne j$ and $\pm e_i$. The center of $Sp(2r)$ is $\mathbb{Z}_2$ with the non-unit element $z = \mathrm{e}^{-2\pi \mathrm{i}\theta}$ for $\theta = \lambda_r^\vee = (1/2)\sum_{i=1}^r e_i$. The permutation $zi = r - i$ for $i = 0, 1, \ldots, r$ generates a symmetry of the extended Dynkin diagram:



Group $Sp(2r)$ is simply connected. The adjoint action of w_z on the Cartan algebra is given by

$$w_z e_i w_z^{-1} = -e_{r-i+1}. \tag{4.16}$$

It may be generated by the product:

$$r_{\alpha_r} r_{\alpha_{r-1}} r_{\alpha_r} \cdots r_{\alpha_2} \cdots r_{\alpha_{r-1}} r_{\alpha_r} r_{\alpha_1} \cdots r_{\alpha_{r-1}} r_{\alpha_r} \tag{4.17}$$

of $r(r+1)/2$ reflections in simple roots. Element w_z may be taken as the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & \cdots & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \cdots & \mathrm{i} & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & \mathrm{i} & 0 & \cdots & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{4.18}$$

in $Sp(2r)$. Setting also $w_1 = 1$, we infer that

$$c_{1,1} = c_{1,z} = c_{z,1} = 1, \qquad c_{z,z} = w_z^2 = -1 = z \tag{4.19}$$

so that we may take

$$b_{1,1} = b_{1,z} = b_{z,1} = 0, \qquad b_{z,z} = \theta \tag{4.20}$$

which results in

$$U_{z^n, z^{n'}, z^{n''}} = \begin{cases} 1 & \text{for } (n, n', n'') \neq (1, 1, 1), \\ (-1)^r & \text{for } n = n' = n'' = 1. \end{cases} \tag{4.21}$$

For r odd, the cocycle U is cohomologically non-trivial. As a result

$$\kappa = \begin{cases} 1 & \text{for } r \text{ even}, \\ 2 & \text{for } r \text{ odd} \end{cases} \tag{4.22}$$

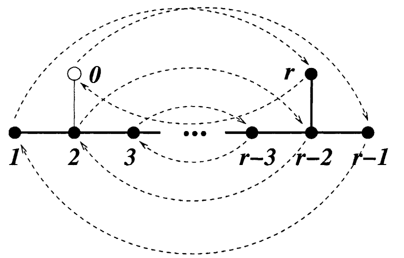
and for those values one may take $u_{z^n, z^{n'}} \equiv 1$ as the solution of Eq. (3.28).

4.4. Groups $D_r = \mathrm{Spin}(2r)$, $r = 3, 4, \ldots$

For $r = 3$, $\mathrm{Spin}(6) \cong SU(4)$. The Lie algebra of group D_r is $so(2r)$ composed of imaginary antisymmetric $(2r) \times (2r)$ matrices. The Cartan algebra may be taken as composed of r blocks $\begin{pmatrix} 0 & -\mathrm{i}t_i \\ \mathrm{i}t_i & 0 \end{pmatrix}$ placed diagonally. In particular, let e_i denote the matrix corresponding to $t_j = \delta_{ij}$. With the invariant form normalized so that $\mathrm{tr}\, e_i e_j = \delta_{ij}$, roots and coroots of $so(2r)$ have the form $\pm e_i \pm e_j$ for $i \neq j$. The simple roots may be chosen as $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, r-1$ and $\alpha_r = e_{r-1} + e_r$.

4.5. Case of r odd

Here the center is $\mathbb{Z}_4$ and it may be generated by $z = \mathrm{e}^{-2\pi \mathrm{i}\theta}$ with $\theta = \lambda_r^\vee = (1/2)\sum_{i=1}^r e_i$. The permutation $z0 = r-1$, $z1 = r$, $zi = r-i$ for $i = 2, \ldots, r-2$, $z(r-1) = 1$, $zr = 0$ induces the extended Dynkin diagram symmetry (for $r \geq 5$):



$SO(2r) = \mathrm{Spin}(2r)/\{1, z^2\}$. The adjoint action of w_z on the Cartan algebra is given by

$$w_z e_i w_z^{-1} = \begin{cases} e_r & \text{for } i = 1, \\ -e_{r-i+1} & \text{for } i \neq 1. \end{cases} \tag{4.23}$$

It may be generated by the product:

$$r_{\alpha_{r-1}} r_{\alpha_{r-2}} r_{\alpha_r} \cdots r_{\alpha_4} \cdots r_{\alpha_{r-1}} r_{\alpha_3} \cdots r_{\alpha_{r-2}} r_{\alpha_r} r_{\alpha_2} \cdots r_{\alpha_{r-1}} r_{\alpha_1} \cdots r_{\alpha_{r-2}} r_{\alpha_r} \tag{4.24}$$

of $(r(r-1))/2$ reflections in simple roots. Element w_z may be taken as a lift to $\mathrm{Spin}(2r)$ of the matrix:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \iddots & & \\ 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{4.25}$$

in $SO(2r)$. We shall take $w_{z^n} = w_z^n$ for $n = 0, 1, 2, 3$. Then

$$c_{z^n, z^m} = \begin{cases} 1 & \text{if } n + m < 4, \\ w_z^4 & \text{if } n + m \geq 4. \end{cases} \tag{4.26}$$

It suffices then to determine the value of w_z^4. Since this element projects to identity in $SO(2r)$, it is either equal to 1 or to z^2. To determine which is the case, note that we may set $w_z = \mathcal{O}\,\mathrm{e}^{2\pi\mathrm{i}X}\mathcal{O}^{-1}$, where $\mathcal{O}$ is an element of Spin$(2r)$ projecting to the matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ 0 & \sqrt{2} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{4.27}$$

and $X = (1/4)e_1 + (1/2)(e_{(r+3)/2} + \cdots + e_r)$ so that $\mathrm{e}^{2\pi\mathrm{i}X}$ projects to the matrix:

$$\mathrm{e}^{2\pi\mathrm{i}X} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \tag{4.28}$$

in $SO(2r)$. It follows that $w_z^4 = \mathrm{e}^{8\pi\mathrm{i}X} = z^2$ since $4X$ is not in the coroot lattice. We may take

$$b_{z^n,z^m} = \begin{cases} 0 & \text{if } n+m < 4, \\ 2\theta & \text{if } n+m \geq 4. \end{cases} \tag{4.29}$$

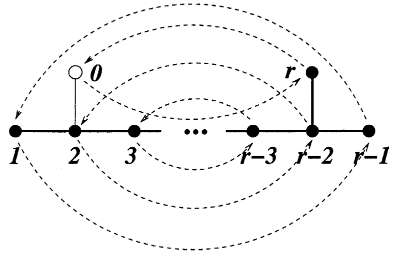
This results in

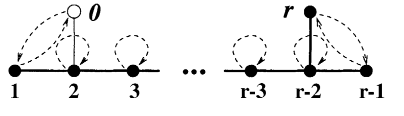
$$U_{z^n,z^{n'},z^{n''}} = (-1)^{n''(n+n'-[n+n'])/4} \tag{4.30}$$

for $0 \leq n, n', n'' \leq 3$, where now $[m] = m \bmod 4$ with $0 \leq [m] \leq 3$. If $Z = \mathbb{Z}_4$ then U is cohomologically non-trivial, hence $\kappa = 2$ in this case. On the other hand, the cocycle (4.30) becomes trivial when restricted to the cyclic subgroup of order 2 generated by z^2 so that $\kappa = 1$ if $Z = \mathbb{Z}_2$. In both cases, for the above values of κ, one may take $u_{z^n, z^{n'}} \equiv 1$ as the solution of Eq. (3.28).

4.6. Case of r even

Here the center is $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is generated by $z_1 = \mathrm{e}^{-2\pi\mathrm{i}\theta_1}$ and $z_2 = \mathrm{e}^{-2\pi\mathrm{i}\theta_2}$ for $\theta_1 = \lambda_r^\vee = (1/2)\left(\sum_{i=1}^r e_i\right)$ and $\theta_2 = \lambda_1^\vee = e_1$. These elements induce the permutations $z_1 0 = r$, $z_1 i = r - i$ for $i = 1, \ldots, r-1$, $z_1 r = 0$, $z_2 0 = 1$, $z_2 1 = 0$, $z_2 i = i$ for $i = 2, \ldots, r-2$, $z_2(r-1) = r$, $z_2 r = r - 1$ giving rise to the symmetries of the extended Dynkin diagrams:





$SO(2r) = \mathrm{Spin}(2r)/\{1, z_2\}$. The adjoint actions of w_{z_1} and w_{z_2} on the Cartan algebra are given by

$$w_{z_1} e_i w_{z_1}^{-1} = -e_{r-i+1}, \qquad w_{z_2} e_i w_{z_2}^{-1} = \begin{cases} -e_i & \text{for } i = 1, r, \\ e_i & \text{for } i \neq 1, r. \end{cases} \tag{4.31}$$

They may be generated by the products:

$$r_{\alpha_r} \cdots r_{\alpha_4} \cdots r_{\alpha_{r-1}} r_{\alpha_3} \cdots r_{\alpha_{r-2}} r_{\alpha_r} r_{\alpha_2} \cdots r_{\alpha_{r-1}} r_{\alpha_1} \cdots r_{\alpha_{r-2}} r_{\alpha_r} \tag{4.32}$$

and

$$r_{\alpha_1} \cdots r_{\alpha_{r-2}} r_{\alpha_r} r_{\alpha_{r-1}} \cdots r_{\alpha_2} r_{\alpha_1} \tag{4.33}$$

of, respectively, $r(r-1)/2$ and $2(r-1)$ reflections in simple roots. Elements w_{z_1} and w_{z_2} may be taken as lifts to $\mathrm{Spin}(2r)$ of the $SO(2r)$ matrices:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix}, \tag{4.34}$$

respectively. We may set

$$w_{z_1} = \mathcal{O}_1\, \mathrm{e}^{2\pi \mathrm{i} X_1} \mathcal{O}_1^{-1}, \qquad w_{z_2} = \mathcal{O}_2\, \mathrm{e}^{2\pi \mathrm{i} X_2} \mathcal{O}_2^{-1} = \mathcal{O}_1 \mathcal{O}_2\, \mathrm{e}^{2\pi \mathrm{i} X_2} \mathcal{O}_2^{-1} \mathcal{O}_1^{-1}, \tag{4.35}$$

where $\mathcal{O}_i$ are in Spin$(2r)$ and project to the $SO(2r)$ matrices:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & \cdots & -1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{4.36}$$

respectively, with $X_1 = (1/2)(e_{(r/2)+1} + \cdots + e_r)$ and $X_2 = (1/2)e_1$. The exponentials $\mathrm{e}^{2\pi \mathrm{i} X_1}$ and $\mathrm{e}^{2\pi \mathrm{i} X_2}$ project in turn to the matrices:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{4.37}$$

respectively. Since $w_{z_i}^2$ projects to 1 in $SO(2r)$ it is equal to 1 or to z_2 in Spin$(2r)$. Which is the case, depends on whether $2X_i$ is in the coroot lattice. We infer that

$$w_{z_1}^2 = \begin{cases} 1 & \text{if } r \text{ is divisible by } 4, \\ z_2 & \text{otherwise}, \end{cases} \qquad w_{z_2}^2 = z_2. \tag{4.38}$$

Besides,

$$\begin{aligned} w_{z_1} w_{z_2} w_{z_1}^{-1} w_{z_2}^{-1} &= \mathcal{O}_1 (\mathrm{e}^{2\pi \mathrm{i} X_1} \mathcal{O}_2\, \mathrm{e}^{2\pi \mathrm{i} X_2} \mathcal{O}_2^{-1}\, \mathrm{e}^{-2\pi \mathrm{i} X_1} \mathcal{O}_2\, \mathrm{e}^{-2\pi \mathrm{i} X_2} \mathcal{O}_2^{-1}) \mathcal{O}_1^{-1} \\ &= \mathcal{O}_1 (\mathrm{e}^{2\pi \mathrm{i} X_1} w_{z_2}\, \mathrm{e}^{-2\pi \mathrm{i} X_1} w_{z_2}^{-1}) \mathcal{O}_1^{-1} = \mathcal{O}_1\, \mathrm{e}^{2\pi \mathrm{i} e_r} \mathcal{O}_1^{-1} = z_2. \end{aligned} \tag{4.39}$$

Setting $w_1 = 1$ and $w_{z_1 z_2} = w_{z_1} w_{z_2}$, we infer that for r divisible by 4,

$$c_{z,z'} = \begin{cases} z_2 & \text{if } (z,z') = (z_2,z_1), (z_2,z_2), (z_1z_2,z_1), (z_1z_2,z_2), \\ 1 & \text{otherwise} \end{cases} \tag{4.40}$$

and for r not divisible by 4,

$$c_{z,z'} = \begin{cases} z_2 & \text{if } (z,z') = (z_1,z_1), (z_1,z_1z_2), (z_2,z_1), (z_2,z_2), (z_1z_2,z_2), (z_1z_2,z_1z_2), \\ 1 & \text{otherwise.} \end{cases} \tag{4.41}$$

We may then take for r divisible by 4,

$$b_{z,z'} = \begin{cases} \theta_2 & \text{if } (z,z') = (z_2,z_1), (z_2,z_2), (z_1z_2,z_1), (z_1z_2,z_2), \\ 0 & \text{otherwise,} \end{cases} \tag{4.42}$$

and for r not divisible by 4,

$$b_{z,z'} = \begin{cases} \theta_2 & \text{if } (z,z') = (z_1,z_1), (z_1,z_1z_2), (z_2,z_1), (z_2,z_2), (z_1z_2,z_2), (z_1z_2,z_1z_2), \\ 0 & \text{otherwise.} \end{cases} \tag{4.43}$$

Explicit calculation gives

$$U_{z,z',z''} = \begin{cases} (-1)^{1+r/2} & \text{for } (z,z',z'') = (z_1z_2,z_1,z_1), (z_1z_2,z_1,z_1z_2), \\ (-1)^{r/2} & \text{for } (z,z',z'') = (z_1,z_1,z_1), (z_1,z_1,z_1z_2), (z_1,z_1z_2,z_1), \\ & \quad (z_1,z_1z_2,z_1z_2), (z_1z_2,z_1z_2,z_1), (z_1z_2,z_1z_2,z_1z_2), \\ -1 & \text{for } (z,z',z'') = (z_2,z_1,z_1), (z_2,z_1,z_1z_2), (z_2,z_2,z_1), \\ & \quad (z_2,z_2,z_1z_2), (z_1z_2,z_2,z_1), (z_1z_2,z_2,z_1z_2), \\ 1 & \text{otherwise.} \end{cases} \tag{4.44}$$

The cocycle U^{K} is cohomologically non-trivial if $\kappa r/2$ is odd. If κ is even, it is trivial, and any 2-cocycle u solves Eq. (3.28). In particular, we may take

$$u_{z,z'} = \begin{cases} \pm 1 & \text{for } (z,z') = (z_2,z_1), (z_2,z_1z_2), (z_1z_2,z_1), (z_1z_2,z_1z_2), \\ 1 & \text{otherwise} \end{cases} \tag{4.45}$$

representing two non-equivalent classes in $H^2(Z, U(1))$. When κ is odd and $r/2$ is even then U^{K} is cohomologically trivial and

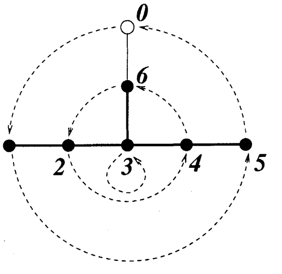
$$u_{z,z'} = \begin{cases} \pm \mathrm{i} & \text{for } (z,z') = (z_2,z_1), (z_2,z_1z_2), (z_1z_2,z_1), (z_1z_2,z_1z_2), \\ 1 & \text{otherwise} \end{cases} \tag{4.46}$$

give two solutions of Eq. (3.28) differing by a non-trivial cocycle (4.45). Hence for $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\kappa = 1$ if $r/2$ is even and $\kappa = 2$ for $r/2$ odd.

If Z is the $\mathbb{Z}_2$ subgroup generated by z_1 or by $z_1 z_2$ then the restriction of U^{K} to Z is cohomologically non-trivial if $\kappa r/2$ is odd and is trivial if $\kappa r/2$ is even. Hence $\kappa = 1$ if $r/2$ is even and $\kappa = 2$ if it is odd. For $Z = \mathbb{Z}_2$ generated by z_2, the restriction of U to Z is trivial so that $\kappa = 1$. One may take $u_{z,z'} \equiv 1$ as the solution of Eq. (3.28) in those cases.

4.7. Group E_6

We shall identify the Cartan algebra of the exceptional group E_6 with the subspace of $\mathbb{R}^7$ with the first six coordinates summing to zero, with the scalar product inherited from $\mathbb{R}^7$. The simple roots, may be taken as $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, 5$ and $\alpha_6 = (1/2)(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6) + (1/\sqrt{2})e_7$, where e_i are the vectors of the canonical bases of $\mathbb{R}^7$. The center of E_6 is $\mathbb{Z}_3$ and it is generated by $z = \mathrm{e}^{-2\pi \mathrm{i}\theta}$ with $\theta = \lambda_5^\vee = (1/6)(e_1 + e_2 + e_3 + e_4 + e_5 - 5e_6) + (1/\sqrt{2})e_7$. The permutation $z0 = 1$, $z1 = 5$, $z2 = 4$, $z3 = 3$, $z4 = 6$, $z5 = 0$, $z6 = 2$ induces the symmetry of the extended Dynkin diagram:



The adjoint action of w_z on the Cartan algebra may be generated by setting

$$
\begin{aligned}
&w_z e_1 w_z^{-1} = -e_6, \qquad w_z e_2 w_z^{-1} = -e_5, \qquad w_z e_3 w_z^{-1} = -e_4,\\
&w_z e_4 w_z^{-1} = -e_3, \qquad w_z e_5 w_z^{-1} = \frac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6) - \frac{1}{\sqrt{2}} e_7,\\
&w_z e_6 w_z^{-1} = \frac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6) + \frac{1}{\sqrt{2}} e_7,\\
&w_z e_7 w_z^{-1} = \frac{1}{\sqrt{2}}(-e_1 + e_2)
\end{aligned}
\tag{4.47}
$$

and is given by the product:

$$
r_{\alpha_1} r_{\alpha_2} r_{\alpha_3} r_{\alpha_4} r_{\alpha_5} r_{\alpha_6} r_{\alpha_3} r_{\alpha_2} r_{\alpha_1} r_{\alpha_4} r_{\alpha_3} r_{\alpha_2} r_{\alpha_6} r_{\alpha_3} r_{\alpha_4} r_{\alpha_5}
\tag{4.48}
$$

of 16 reflections that may be rewritten as the product of 4 reflections $r_{\beta_1} r_{\beta_4} r_{\beta_5} r_{\beta_2}$ in non-simple roots:

$$
\begin{aligned}
&\beta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \qquad \beta_2 = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,\\
&\beta_4 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6, \qquad \beta_5 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,
\end{aligned}
\tag{4.49}
$$

The family of roots $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ with

$$
\beta_3 = -\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, \qquad \beta_6 = \alpha_3
\tag{4.50}
$$

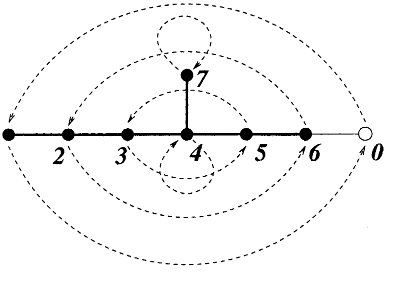
provides another set of simple roots for E_6 corresponding to the same Cartan matrix. The roots β_i with $i \leq 5$ and their step generators $e_{\pm\beta_i}$ generate an A_5 subalgebra of E_6 which, upon exponentiation, gives rise to an $SU(6)$ subgroup of group E_6. The group elements that implement by conjugation the Weyl reflections r_{β_i} of the Cartan algebra of E_6 may be taken as $\mathrm{e}^{\pi/2\mathrm{i}}(e_{\beta_i} + e_{-\beta_i})$ so that they belong to the $SU(6)$ subgroup for $i \leq 5$. We infer that, identifying roots β_i for $i \leq 5$ with the standard roots of A_5, the element w_z may be taken as the matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in SU(6) \subset E_6 \tag{4.51}$$

which satisfies $w_z^3 = 1$. Setting now $w_1 = 1$ and $w_{z^2} = w_z^2$, we end up with the trivial cocycle $c_{z,z'}$ and may take $b_{z,z'} \equiv 0$. Consequently, the cocycle U is also trivial and $\kappa = 1$, $u_{z^n,z^{n'}} \equiv 1$ solve Eq. (3.28).

4.8. Group E_7

The Cartan algebra of E_7 may be identified with the subspace of $\mathbb{R}^8$ orthogonal to the vector $(1, 1, \ldots, 1)$ with the simple roots $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, 6$ and $\alpha_7 = (1/2)(-e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 + e_8)$ with e_i the vectors of the canonical basis of $\mathbb{R}^8$. The center of E_7 is $\mathbb{Z}_2$ with the non-unit element $z = \mathrm{e}^{-2\pi\mathrm{i}\theta}$ for $\theta = \lambda_1^\vee = (1/4)(3, -1, -1, -1, -1, -1, -1, 3)$. The permutation $z0 = 1$, $z1 = 0$, $z2 = 6$, $z3 = 5$, $z4 = 4$, $z5 = 3$, $z6 = 2$, $z7 = 7$ generates the symmetry of the extended Dynkin diagram:



The adjoint action of w_z may by obtained by setting

$$w_z e_i w_z^{-1} = -e_{9-i} \tag{4.52}$$

and is given by the product:

$$r_{\alpha_1}r_{\alpha_2}r_{\alpha_3}r_{\alpha_4}r_{\alpha_5}r_{\alpha_7}r_{\alpha_4}r_{\alpha_6}r_{\alpha_3}r_{\alpha_5}r_{\alpha_2}r_{\alpha_4}r_{\alpha_1}r_{\alpha_3}r_{\alpha_7}r_{\alpha_4}r_{\alpha_2}r_{\alpha_5}r_{\alpha_3}r_{\alpha_6}r_{\alpha_4}r_{\alpha_7}r_{\alpha_5}r_{\alpha_4}r_{\alpha_3}r_{\alpha_2}r_{\alpha_1} \tag{4.53}$$

of 27 simple root reflections that may be rewritten as the product of three reflections $r_{\beta_1}r_{\beta_3}r_{\beta_7}$ for

$$\begin{aligned}\beta_1 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7 = \omega(\alpha_1),\\ \beta_3 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = \omega(\alpha_3),\\ \beta_7 &= \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 = \omega(\alpha_7),\end{aligned} \tag{4.54}$$

where $\omega = r_{\alpha_1}r_{\alpha_2}r_{\alpha_3}r_{\alpha_4}r_{\alpha_5}r_{\alpha_7}r_{\alpha_4}r_{\alpha_6}r_{\alpha_3}r_{\alpha_5}r_{\alpha_2}r_{\alpha_4}$. The roots β_1, β_3 and β_7 may be completed to a new system of simple roots of E_7 by setting

$$\begin{aligned}\beta_2 &= -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7) = \omega(\alpha_2),\\ \beta_4 &= -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) = \omega(\alpha_4),\\ \beta_6 &= \alpha_7 = \omega(\alpha_6).\end{aligned} \tag{4.55}$$

In particular, $\beta_1, \beta_2, \beta_3, \beta_4, \beta_7$ and their step generators span a subalgebra $A_5 \subset E_7$ that, upon exponentiation, gives rise to a subgroup $SU(6)$ in group E_7. The element w_z implementing by conjugation the Weyl transformation (4.53) may be chosen as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \in SU(6) \subset E_7 \tag{4.56}$$

upon identifying of the roots $\beta_1, \beta_2, \beta_3, \beta_4, \beta_7$ with the standard roots of $A_5 = su(6)$. In particular, $w_z^2 = -1 \in SU(6)$ or

$$w_z^2 = \mathrm{e}^{\pi\mathrm{i}(\beta_1+\beta_3+\beta_7)} = (-\mathrm{i}, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}) = \mathrm{e}^{2\pi\mathrm{i}\theta}. \tag{4.57}$$

With that choice of w_z, we infer that

$$c_{1,1} = c_{1,z} = c_{z,1} = 1, \qquad c_{z,z} = w_z^2 = \mathrm{e}^{2\pi\mathrm{i}\,\theta} \tag{4.58}$$

and we may take

$$b_{1,1} = b_{1,z} = b_{z,1} = 0, \qquad b_{z,z} = \theta. \tag{4.59}$$

This leads to

$$U_{z^n,z^{n'},z^{n''}} = \begin{cases} 1 & \text{for } (n, n', n'') \neq (1, 1, 1),\\ -1 & \text{for } n = n' = n'' = 1. \end{cases} \tag{4.60}$$

U^{K} is trivial if κ is even and is cohomologically non-trivial when κ is odd. Hence $\kappa = 2$ and one may take $u_{z^n,z^{n'}} \equiv 1$ as the solution of Eq. (3.28) for that value of κ.

5. Conclusions

We have presented an explicit construction of the basic gerbes over groups $G' = G/Z$ where G is a simple compact connected and simply connected group and Z is a subgroup of the center of G. By definition of the basic gerbe, the pullback to G of its curvature H' is the closed 3-form $H = (\kappa/12\pi)\,\mathrm{tr}\,(g^{-1}\,\mathrm{d}g)^3$ with the level κ taking the lowest possible positive value. The restriction on κ came from the cohomological equation (3.28) that assures the associativity of the gerbe's groupoid product. In agreement with the general theory, see [7,9], the levels κ of the basic gerbes are the lowest positive numbers for which the periods of H' belong to $2\pi\mathbb{Z}$. They have been previously found in Ref. [5] and we have recovered here the same set of numbers. The basic gerbe over G' is unique up to stable isomorphisms except for $G' = SO(4N)/\mathbb{Z}_2$. In the latter case, using the two different choices of sign in the solutions (4.45) or (4.46) of the cohomological relation (3.28), one obtains basic gerbes belonging to two different stable isomorphism classes, the doubling already observed in Ref. [5]. We plan to use the results of the present paper in order to extend the classification of the fully symmetric branes in groups $SU(N)/Z$ worked out in Ref. [9] to all groups G'.

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Appendix A

We shall obtain here the condition (3.26) for the associativity of the groupoid product μ' defined by (3.23). Let $(y, y', y'', y''') \in Y'^{[4]}$ with $(y, zy', z(z'y''), z(z'(z''y''')))$ belonging to Y_{ijkl} and projecting to $g \in \mathcal{O}_{ijkl}$. Taking $h \in G$ such that $\rho_{ijkl}(g) = hG_{ijkl}$, we may complete Eqs. (3.17)–(3.19) by

$$z(z'(z''y''')) = (g, h\gamma'''^{-1}), \tag{A.1}$$

$$y''' = ((zz'z'')^{-1}g, hw_z w_{z'} w_{z''}((\gamma'''_z)_{z'})^{-1}_{z''}), \tag{A.2}$$

$$z''y''' = ((zz')^{-1}g, hw_z w_{z'}(\gamma'''_z)^{-1}_{z'}), \tag{A.3}$$

$$(z'z'')y''' = (z^{-1}g, hw_z(c^{-1}_{z',z''}\gamma'''_z)^{-1}), \tag{A.4}$$

$$(zz'z'')y''' = (g, h(c^{-1}_{zz',z''}c^{-1}_{z,z'}\gamma''')^{-1}). \tag{A.5}$$

The G_{ijk}-orbits in Eqs. (3.20)–(3.22) may be now replaced by the G_{ijkl}-orbits. We shall need further line bundle elements. Let

$$\begin{aligned}\ell_{k_{zz'}l_{zz'}} &= (((zz')^{-1}g, hw_z w_{z'}), [(\tilde{\gamma}''_z)_{z'}, (\tilde{\gamma}'''_z)_{z'}, u'']^{\rm K}_{k_{zz'}l_{zz'}})G_{i_{zz'}j_{zz'}k_{zz'}l_{zz'}} \\ &= \chi_{kl}(\tilde{c}_{z,z'})^{\rm K}(((zz')^{-1}g, hw_{zz'}), [(\tilde{c}^{-1}_{z,z,'}\tilde{\gamma}'')_{zz'}, (\tilde{c}^{-1}_{z,z'}\tilde{\gamma}''')_{zz'}, u'']^{\rm K}_{k_{zz'}l_{zz'}}) \\ &\quad \times G_{i_{zz'}j_{zz'}k_{zz'}l_{zz'}} \in L^{\rm K}_{(y'',z''y''')} = L'_{(y'',y''')},\end{aligned} \tag{A.6}$$

where we have used the identifications entering the definition of the line bundle $L_{k_{zz'}\ell_{zz'}}$. Similarly, let

$$\begin{aligned}\ell_{il} &= ((g,h), [\tilde{\gamma}, \tilde{c}^{-1}_{zz',z''}\tilde{c}^{-1}_{z,z'}\tilde{\gamma}''', uu'u'']^{\rm K}_{il})G_{ijkl} \\ &= \chi_l((\delta\tilde{c})_{z,z',z''})^{\rm K}((g,h), [\tilde{\gamma}, \tilde{c}^{-1}_{z,z'z''}(w_z\tilde{c}^{-1}_{z',z''}w_z^{-1})\tilde{\gamma}''', uu'u'']^{\rm K}_{il})G_{ijkl} \in L^{\rm K}_{(y,(zz'z'')y''')} \\ &= L'_{(y,y''')},\end{aligned} \tag{A.7}$$

where the 3-cocycle $\delta\tilde{c}$ is given by (3.10). Finally, let

$$\ell_{j_zl_z} = ((z^{-1}g, hw_z), [\tilde{\gamma}'_z, \tilde{c}^{-1}_{z',z''}\tilde{\gamma}'''_z, u'u'']^{\rm K}_{j_zl_z})G_{i_zj_zk_zl_z} \in L^{\rm K}_{(y',(z'z'')y''')} = L'_{(y',y''')}. \tag{A.8}$$

Now

$$\mu'(\mu'(\ell_{ij}\otimes\ell_{j_zk_z})\otimes\ell_{k_{zz'}l_{zz'}}) = u^{ijk}_{z,z'}\mu'(\ell_{ik}\otimes\ell_{k_{zz'}l_{zz'}}) = u^{ijk}_{z,z'}u^{ikl}_{zz',z''}\chi_{kl}(\tilde{c}_{z,z'})^{\rm K}\ell_{il}. \tag{A.9}$$

On the other hand,

$$\begin{aligned}&\mu'(\ell_{ij}\otimes\mu'(\ell_{j_zk_z}\otimes\ell_{k_{zz'}l_{zz'}})) \\ &= u^{j_zk_zl_z}_{z',z''}\mu'(\ell_{ij}\otimes\ell_{j_zl_z}) = u^{j_zk_zl_z}_{z',z''}u^{ijl}_{z,z'z''}\chi_l((\delta\tilde{c})_{z,z',z''})^{-\rm K}\ell_{il}.\end{aligned} \tag{A.10}$$

Equating both expressions we infer condition (3.26).

Appendix B

Proof of Lemma 3. With $(u^{ijk}_{z,z'})$ given by Eq. (3.32) and $(u_{z,z'})$ solving Eq. (3.28), the left-hand side of (3.26) becomes

$$\begin{aligned}&\chi_{l_z(z'z''z''0)}(\tilde{c}_{z',z''})^{-\rm K}\chi_{l(zz'z''0)}(\tilde{c}_{zz',z''})^{\rm K}\chi_{l(zz'z''0)}(\tilde{c}_{z,z'z''})^{-\rm K} \\ &\chi_{k(zz'0)}(\tilde{c}_{z,z'})^{\rm K}\chi_{(zz'0)(zz'z''0)}(\tilde{c}_{z,z'})^{\rm K}\chi_{zz'z''0}((\delta\tilde{c})_{z,z',z''})^{\rm K}.\end{aligned} \tag{B.1}$$

The first factor may be rewritten as $\chi_{l(zz'z''0)}(w_z(\tilde{c}_{z',z''})w_z^{-1})^{-\rm K}$ using the 2nd identity in (3.24) and combines with the next two to

$$\begin{aligned}&\chi_{l(zz'z''0)}((\delta\tilde{c})_{z,z',z''})^{-\rm K}\chi_{l(zz'z''0)}(\tilde{c}_{z,z'})^{-\rm K} \\ &= \chi_l((\delta\tilde{c})_{z,z',z''})^{\rm K}\chi_{zz'z''0}((\delta\tilde{c})_{z,z',z''})^{-\rm K}\chi_{l(zz'z''0)}(\tilde{c}_{z,z'})^{-\rm K}.\end{aligned} \tag{B.2}$$

With the next three factors, it reproduces with the use of property (2.19) the right-hand side of (3.26). □

Proof of Lemma 2. This proceeds similarly. With the use of the explicit expression (3.29), the middle term of (3.31) becomes

$$\begin{aligned}&\chi_{(z'z''0)(z'z''z'''0)}(\tilde{c}_{z',z''})\,\chi_{(zz'z''0)(zz'z''z'''0)}(\tilde{c}_{zz',z''})^{-1}\,\chi_{(zz'z''0)(zz'z''z'''0)}(\tilde{c}_{z,z'z''})\\&\chi_{(zz'0)(zz'z''z'''0)}(\tilde{c}_{z,z'})^{-1}\,\chi_{(zz'0)(zz'z''0)}(\tilde{c}_{z,z'})\,\chi_{z'z''z'''0}((\delta\tilde{c})_{z',z'',z'''})\\&\chi_{zz'z''z'''0}((\delta\tilde{c})_{zz',z'',z'''})^{-1}\,\chi_{zz'z''z'''0}((\delta\tilde{c})_{z,z'z'',z'''})\\&\chi_{zz'z''z'''0}((\delta\tilde{c})_{z,z',z''z'''})^{-1}\,\chi_{zz'z''0}((\delta\tilde{c})_{z,z',z''}).\end{aligned}\tag{B.3}$$

The first factor is equal to $\chi_{(zz'z''0)(zz'z''z'''0)}(w_z\tilde{c}_{z',z''}w_z^{-1})$, see (3.24), and it combines with the next four ones to

$$\chi_{(zz'z''0)(zz'z''z'''0)}((\delta\tilde{c})_{z,z',z''}) = \chi_{zz'z''0}((\delta\tilde{c})_{z,z',z''})^{-1}\chi_{zz'z''z'''0}((\delta\tilde{c})_{z,z',z''}),\tag{B.4}$$

see (2.19) and (2.21). Together with the remaining factors, one obtains, rewriting the sixth factor as $\chi_{zz'z''z'''0}(w_z(\delta\tilde{c}_{z',z'',z'''})w_z^{-1})$, an expression that reduces to

$$\chi_{zz'z''z'''0}((\delta^2\tilde{c})_{z,z',z'',z'''})\tag{B.5}$$

and is equal to 1 due to the triviality of δ^2.

As for the independence of the cohomology class $[U] \in H^3(Z, U(1))$ of the choice of $b_{z,z'} \in \mathbf{t}$, a simple algebra with the use of Eqs. (2.10), (2.20) and (3.24) shows that under the replacement (3.6), the cocycle U changes to $U\delta u$ for

$$u_{z,z'} = \chi_{ij}(\mathrm{e}_{ij}^{2\pi\mathrm{i}a_z})\chi_j(\mathrm{e}_j^{2\pi\mathrm{i}q_{z,z'}})\tag{B.6}$$

with $i = z0$ and $j = zz'0$. □

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